

UNIFORMLY BOUNDED ORTHONORMAL POLYNOMIALS ON THE SPHERE

JORDI MARZO AND JOAQUIM ORTEGA-CERDÀ

ABSTRACT. Given any $\varepsilon > 0$, we construct an orthonormal system of n_k uniformly bounded polynomials of degree at most k on the unit sphere in \mathbb{R}^{m+1} where n_k is bigger than $1 - \varepsilon$ times the dimension of the space of polynomials of degree at most k . Similarly we construct an orthonormal system of sections of powers L^k of a positive holomorphic line bundle on a compact Kähler manifold with cardinality bigger than $1 - \varepsilon$ times the dimension of the space of global holomorphic sections to L^k .

1. INTRODUCTION

In [15], the authors construct what are now known as Ryll-Wojtaszczyk polynomials. These are the elements of a sequence $\{W_k\}_{k \geq 1}$ of homogeneous polynomials of degree $k \geq 1$ in n complex variables such that they are of L^2 -norm one on the unit sphere in \mathbb{C}^n and uniformly bounded there. These polynomials have proved to be very useful to construct functions with precise growth restrictions and most notably they can be used to construct inner functions in the unit ball in several variables, see [1]. For a beautiful monograph about the construction of inner functions in several variables and related problems we refer to [14].

The existence of inner functions in the unit ball \mathbb{B}_m of \mathbb{C}^m , for $m > 1$, was an open problem for many years. As inner functions are (up to multiplication by constants) those $f \in H^2(\mathbb{C}^m)$ such that $\|f\|_\infty = \|f\|_2$, it was natural to try to find first uniformly bounded sequences of polynomials. In the unit ball of $\mathbb{B}_2 \subset \mathbb{C}^2$, Bourgain, in [3], found a uniformly bounded orthonormal basis of $H^2(\mathbb{B}_2)$ by constructing a sequence of bounded orthonormal bases of the spaces of holomorphic homogeneous polynomials in \mathbb{B}_2 . The same question in higher dimensions remains open. Specifically, it is not known if there exist uniformly bounded orthonormal bases for the spaces of holomorphic homogeneous polynomials in \mathbb{B}_m , for $m > 2$. We observe that the existence of Ryll-Wojtaszczyk polynomials, i.e. just one bounded polynomial for each degree, implies that $H^2(\mathbb{B}_m)$ do have a uniformly bounded orthonormal basis formed by polynomials for any $m > 2$, the idea of the construction is due to Olevskii [11, Chap. 4]. See also [14, Appendix I] where a better bound (independent of the dimension) for the elements of the basis is obtained by using powers of an inner function instead of Ryll-Wojtaszczyk polynomials.

The space of homogeneous holomorphic polynomials of degree k in \mathbb{C}^m can be identified with the space $H^0(\mathbb{CP}^{m-1}, L^k)$ of global holomorphic sections of the k -power of the hyperplane bundle $L \rightarrow \mathbb{CP}^{m-1}$ which is endowed with the Fubini-Study metric, so that the L^2 norm of

Date: November 7, 2014.

Key words and phrases. Bounded polynomials, Fekete points.

The authors are supported by the Generalitat de Catalunya (project 2014 SGR 289) and the Spanish Ministerio de Economía y Competitividad (project MTM2011-27932-C02-01).

the section is the same as the $H^2(\mathbb{B}_m)$ norm of the polynomial. Thus, it is possible to consider the same problem of existence of bounded orthonormal basis in a more general setting. In [16], Shiffman constructs a uniformly bounded orthonormal system of sections of powers L^k of a positive holomorphic line bundle over a compact Kähler manifold M . He proves that the number n_k of sections in the orthonormal system is at least $\beta \dim H^0(M, L^k)$, where $0 < \beta < 1$ is a number that depends only on the dimension of M . These orthonormal sections are built in [16] by using linear combinations of reproducing kernels peaking at points situated in a lattice-like structure on the manifold. In the same paper Shiffman raises the question whether using reproducing kernels peaking at the Fekete points one may increase the size of the uniformly bounded orthonormal system of sections. We provide a positive answer to this question.

We proceed as follows: It is known that an arbitrarily small perturbation of an array of Fekete points gives an interpolating sequence, see [13, 8]. Then we use Jaffard's theorem on "well localized" matrices, together with the interpolation property, to deduce that the inverse of the Gramian matrix defined through the kernels defines a bounded operator in ℓ^∞ . Finally we construct the bounded orthonormal sections by following the same arguments as in [16].

One of our main ingredients is that, given $\varepsilon > 0$, it is possible to find interpolating sequences for $H^0(M, L^k)$ with cardinality $(1 - \varepsilon) \dim H^0(M, L^k)$. It is also known, see [8], that there are no (uniform) Riesz basis of reproducing kernels in the space of sections of $H^0(M, L^k)$. Thus this approach cannot provide uniformly bounded orthonormal basis of sections in $H^0(M, L^k)$ which would be the ultimate goal.

We will not only consider the complex manifolds setting (as in [16]) but we deal also with a real variant of the problem. In particular we consider spaces generated by eigenfunctions for the Laplace-Beltrami operator in compact two-point homogeneous Riemannian manifolds. The main example is the sphere \mathbb{S}^m in \mathbb{R}^{m+1} and the corresponding spaces of polynomials of degree at most k . Our aim is to construct many uniformly bounded orthonormal polynomials of degree at most k in $m + 1$ variables restricted to a sphere in \mathbb{R}^{m+1} . We observe that there are no orthonormal basis of reproducing kernels for the space of polynomials of degree at most k , as this would be equivalent to the existence of tight spherical $2k$ -design, [2]. It is not known if there are (uniform) Riesz basis of reproducing kernels, [9]. In any case, as before, our approach cannot provide uniformly bounded orthonormal basis.

The proof in this real setting has one extra difficulty when compared to the positively curved holomorphic line bundle setting, because in the real setting the off-diagonal decay of the corresponding reproducing kernel is not fast enough to make the same argument work and some changes are needed. Thus we prefer to present the proof of the more delicate problem, i.e. the Riemannian setting, and we will point out along the way which are the relevant changes to make in the complex setting.

2. MAIN RESULTS

Our result in the complex manifold setting reads as:

Theorem 2.1. *Let $L \rightarrow M$ be a Hermitian holomorphic line bundle over a compact Kähler manifold M with positive curvature. Then for any $\varepsilon > 0$, there is a constant C_ε such that for any*

$k \in \mathbb{Z}^+$, we can find orthonormal holomorphic sections:

$$s_1^k, \dots, s_{n_k}^k \in H^0(M, L^k), \quad n_k \geq (1 - \varepsilon) \dim H^0(M, L^k),$$

such that $\|s_j^k\|_\infty \leq C_\varepsilon$ for $1 \leq j \leq n_k$ and for all $k \in \mathbb{Z}^+$.

For $M = \mathbb{CP}^{m-1}$ and L the hyperplane section bundle $\mathcal{O}(1)$ endowed with the Fubini-Study metric, one can identify $H^0(\mathbb{CP}^{m-1}, L^k)$ with the space $H_k(\mathbb{B}_m)$ of homogeneous holomorphic polynomials of degree k on \mathbb{C}^m . Then the theorem above gives us the following result.

Corollary 2.2. *For all $m, k \geq 1$ and any $\varepsilon > 0$ there is a constant C_ε and a system of orthonormal homogeneous holomorphic polynomials*

$$p_1^k, \dots, p_{n_k}^k \in H_k(\mathbb{B}_m), \quad n_k \geq (1 - \varepsilon) \dim H_k(\mathbb{B}_m),$$

such that $\|p_j^k\|_{L^\infty(\mathbb{S}^{2m-1})} \leq C_\varepsilon$ for $1 \leq j \leq n_k$.

In the real setting we consider compact two-point homogeneous Riemannian manifolds. These spaces are essentially the sphere, the projective spaces over the fields \mathbb{R}, \mathbb{C} and \mathbb{H} and the Cayley plane. We introduce the notation. Let (M, g) be a compact two-point homogeneous Riemannian manifold of dimension $m \geq 2$. Let dV be the volume element. The (discrete) spectrum of the Laplace-Beltrami operator is a sequence of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty,$$

and we consider the corresponding orthonormal basis of eigenfunctions ϕ_i (so we have $\Delta \phi_i = -\lambda_i \phi_i$). Consider the following subspaces of $L^2(M)$:

$$E_L = \text{span}_{\lambda_i \leq L} \{\phi_i\}.$$

We denote $\dim E_L = k_L$. The reproducing kernels of E_L are given by

$$B_L(z, w) = \sum_{i=1}^{k_L} \phi_i(z) \overline{\phi_i(w)}.$$

These are functions defined by the properties that, for any $w \in M$, we have that $B_L(\cdot, w) \in E_L$ and $\langle \phi, B_L(\cdot, w) \rangle = \phi(w)$, when $\phi \in E_L$. It is known that $\|B_L(\cdot, w)\|_{L^2(M)}^2 = B_L(w, w) \sim L^m$ and also $k_L \sim L^m$, see [6]. We denote by $b_L(z, w) = B_L(z, w) / \|B_L(\cdot, w)\|_{L^2(M)}$ the normalized reproducing kernels.

Our main example is the sphere $M = \mathbb{S}^m$, where the eigenfunctions ϕ_i are spherical harmonics and the spaces E_L are the restriction to the sphere of the space of polynomials in \mathbb{R}^{m+1} . In particular, the space of spherical harmonics of degree at most L i.e. the restriction to the sphere \mathbb{S}^m of the polynomials in $m+1$ variables of degree at most L corresponds to $E_{L(L+m-1)}$.

Our result is the following:

Theorem 2.3. *Given $\varepsilon > 0$ and $L \in \mathbb{Z}^+$ there exist $C_\varepsilon > 0$ and a set $\{s_1^L, \dots, s_{n_L}^L\}$ of orthonormal functions in E_L with $n_L \geq (1 - \varepsilon) \dim E_L$ such that $\|s_j^L\|_{L^\infty(M)} \leq C_\varepsilon$, for all $L \in \mathbb{Z}^+$ and $1 \leq j \leq n_L$.*

As we mentioned before, we use Fekete arrays in the construction of the orthonormal functions.

Definition 2.4. Let $\{\psi_1, \dots, \psi_{k_L}\}$ be any basis in E_L . A set of points $x_1^*, \dots, x_{k_L}^* \in M$ such that

$$|\det(\psi_i(x_j^*))_{i,j}| = \max_{x_1, \dots, x_{k_L} \in M} |\det(\psi_i(x_j))_{i,j}|$$

is a Fekete set of points of degree L for M .

Fekete points are well suited points for interpolation formulas and numerical integration. One reason is that the corresponding Lagrange polynomials are bounded by 1. We use the interpolating properties of these Fekete points. In the following lines we provide the definition of interpolating arrays and some related concepts. For any degree L we take m_L points in M

$$\mathcal{Z}(L) = \{z_{L,j} \in M : 1 \leq j \leq m_L\}, \quad L \geq 0,$$

and assume that $m_L \rightarrow \infty$ as $L \rightarrow \infty$. This yields a triangular array of points $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geq 0}$ in M .

Definition 2.5. Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geq 0}$ be a triangular array with $m_L(\leq k_L)$ for all L . We say that \mathcal{Z} is interpolating, if for all arrays $\{c_{L,j}\}_{L \geq 0, 1 \leq j \leq m_L}$ of values such that

$$\sup_{L \geq 0} \frac{1}{k_L} \sum_{j=1}^{m_L} |c_{L,j}|^2 < \infty,$$

there exists a sequence of functions $Q_L \in E_L$ uniformly bounded in $L^2(M)$ such that $Q_L(z_{L,j}) = c_{L,j}$, for $1 \leq j \leq m_L$, and for all L .

An equivalent definition is the following.

The array \mathcal{Z} is interpolating if and only if the normalized reproducing kernel of E_L at the points $\mathcal{Z}(L)$ form a Riesz sequence i.e.

$$(1) \quad C^{-1} \sum_{j=1}^{m_L} |a_{Lj}|^2 \leq \int_M \left| \sum_{j=1}^{m_L} a_{Lj} b_L(z, z_{L,j}) \right|^2 dV(z) \leq C \sum_{j=1}^{m_L} |a_{Lj}|^2,$$

for any $\{a_{Lj}\}_{L,j}$ with $C > 0$ independent of L .

It is well known also that the interpolating property is equivalent to say that the Gramian matrix

$$G_L = G = (\langle b_L(\cdot, z_{L,i}), b_L(\cdot, z_{L,j}) \rangle)_{i,j} = (L^{-m/2} b_L(z_{L,i}, z_{L,j}))_{i,j}$$

defines a bounded operator in ℓ^2 which is uniformly bounded below, where uniformly means with respect to L , see [4, p. 66].

A nice property of interpolating sequences is that they are uniformly separated. We denote by $d(u, v)$ the geodesic distance between $u, v \in M$.

Definition 2.6. An array $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geq 0}$ is uniformly separated if there is a positive number $\varepsilon > 0$ such that

$$d(z_{L,j}, z_{L,k}) \geq \frac{\varepsilon}{L+1}, \quad \text{if } j \neq k,$$

for all $L \geq 0$.

The right hand side inequality in (1) holds if and only if \mathcal{Z} is uniformly separated, see [9, 13]. The next result provides us with Riesz sequences of reproducing kernels with cardinality almost optimal, see [13, 8] for a proof in the two different settings we are considering.

Theorem 2.7. *Given $L \geq 0$ let $\mathcal{Z}(L)$ be a set of Fekete points of degree L for M . Then, for any $\epsilon > 0$, the array $\{\mathcal{Z}(L_\epsilon)\}_{L \geq 0}$ is interpolating where $L_\epsilon = \lfloor (1 - \epsilon)L \rfloor$.*

The theorem above shows that the normalized reproducing kernels $\{b_L(\cdot, z)\}_{z \in \mathcal{Z}(L_\epsilon)}$ form a Riesz sequence. In the compact complex manifolds setting one can use directly these kernels to continue with the construction of flat sections. Unfortunately, when working with Riemannian manifolds the off-diagonal decay of the reproducing kernels is not fast enough. So we are going to introduce better kernels.

Definition 2.8. *Given $0 < \epsilon \leq 1$ let $\beta_\epsilon : [0, +\infty) \rightarrow [0, 1]$ be a nonincreasing C^∞ function such that $\beta_\epsilon(x) = 1$ for $x \in [0, 1 - \epsilon]$ and $\beta_\epsilon(x) = 0$ if $x > 1$. We consider the following Bochner-Riesz type kernels*

$$B_L^\epsilon(z, w) = \sum_{k=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_k}{L}\right) \phi_k(z) \overline{\phi_k(w)}.$$

In the limiting case, when $\epsilon = 0$, we recover the reproducing kernel for E_L . Observe that one obtains easily, from the corresponding result for the reproducing kernel, that $\|B_L^\epsilon(\cdot, w)\|_2^2 \sim L^m$ for any $w \in M$. The main advantage of these modified kernels is that they have better pointwise estimates than the reproducing kernels. The following was proved in [5, Theorem 2.1]:

$$(2) \quad |B_L^\epsilon(z, w)| \lesssim \frac{L^m}{(1 + Ld(z, w))^N}, \quad z, w \in M$$

where one can take any $N > m$ (changing the constant). The bound for the reproducing kernel is the same than (2) with $N = 1$. As before we denote by lower-case $b_L^\epsilon(z, w)$ the normalized kernel.

Our next result shows that one may replace the reproducing kernels by the Bochner-Riesz type and still get a Riesz sequence.

Lemma 2.9. *Given $\epsilon > 0$ there exist a set of $n_{L,\epsilon}$ points $\{z_j\}_{j=1, \dots, n_{L,\epsilon}}$ with $n_{L,\epsilon} \geq (1 - \epsilon) \dim E_L$ such that the normalized Bochner-Riesz type kernels $\{b_L^\epsilon(\cdot, z_j)\}_{j=1, \dots, n_{L,\epsilon}}$ form a Riesz sequence with uniform bounds i.e.*

$$(3) \quad C^{-1} \sum_{j=1}^{n_{L,\epsilon}} |a_j|^2 \leq \int_M \left| \sum_{j=1}^{n_{L,\epsilon}} a_j b_L^\epsilon(z, z_j) \right|^2 dV(z) \leq C \sum_{j=1}^{n_{L,\epsilon}} |a_j|^2,$$

for any $\{a_j\}_j$ with $C > 0$ independent of L .

Proof. We choose the points z_j for $j = 1, \dots, k_{(1-2\epsilon)L}$ to be a Fekete array in $E_{(1-2\epsilon)L}$. It is clear that by an easy application of Theorem 2.7 (enlarging the space instead of shrinking the set of points) they are an interpolating array for $E_{(1-\epsilon)L}$. We denote $n_{L,\epsilon} = k_{(1-2\epsilon)L}$.

The right hand side inequality in (3) follows essentially from the uniform separation of the sequence. Indeed, let

$$S_{L,\varepsilon}(a) = \sum_{j=1}^{n_{L,\varepsilon}} a_j b_L^\varepsilon(z, z_j).$$

By duality

$$\|S_{L,\varepsilon}(a)\|_2 = \sup_{\|P\|_2=1} |\langle P, S_{L,\varepsilon}(a) \rangle| \sim \frac{1}{L^{m/2}} \sup_{\|P\|_2=1} \left| \sum_{j=1}^{n_{L,\varepsilon}} a_j \int_M B_L^\varepsilon(z_j, w) P(w) dV(w) \right|.$$

The set $\{z_j\}_{j=1,\dots,n_{L,\varepsilon}}$ is uniformly separated and Plancherel-Polya inequality says that

$$\frac{1}{k_L} \sum_{j=1}^{n_{L,\varepsilon}} |\phi(z_j)|^2 \lesssim \|\phi\|_2^2,$$

for all $\phi \in E_L$, see [12, Theorem 4.6.]. Therefore

$$\frac{1}{k_L} \sum_{j=1}^{n_{L,\varepsilon}} \left| \int_M B_L^\varepsilon(z_j, w) P(w) dV(w) \right|^2 \lesssim \left\| \int_M B_L^\varepsilon(\cdot, w) P(w) dV(w) \right\|_2^2 \lesssim \|P\|_2^2.$$

By Cauchy-Schwarz we get the desired inequality.

For the left hand side in (3), let $P_{(1-\varepsilon)L}$ the orthogonal projection from $L^2(\mathbb{S}^d)$ onto $E_{(1-\varepsilon)L}$. Then as $P_{(1-\varepsilon)L}(B_L^\varepsilon(\cdot, w))(z)$ is the reproducing kernel in $E_{(1-\varepsilon)L}$, and z_j for $j = 1, \dots, n_{L,\varepsilon}$ is interpolating for $E_{(1-\varepsilon)L}$

$$\sum_{j=1}^{n_{L,\varepsilon}} |a_j|^2 \lesssim \|P_{[(1-\varepsilon)L]} S_{L,\varepsilon}(a)\|_2^2 \leq \|S_{L,\varepsilon}(a)\|_2^2.$$

□

Given $\varepsilon > 0$, let z_j for $j = 1, \dots, n_{L,\varepsilon}$ be the points given by Lemma 2.9 and let

$$\Delta = (\Delta_{ij})_{i,j=1,\dots,n_{L,\varepsilon}} = \begin{pmatrix} \langle b_L^\varepsilon(\cdot, z_1), b_L^\varepsilon(\cdot, z_1) \rangle & \cdots & \langle b_L^\varepsilon(\cdot, z_1), b_L^\varepsilon(\cdot, z_{n_{L,\varepsilon}}) \rangle \\ \vdots & & \vdots \\ \langle b_L^\varepsilon(\cdot, z_{n_{L,\varepsilon}}), b_L^\varepsilon(\cdot, z_1) \rangle & \cdots & \langle b_L^\varepsilon(\cdot, z_{n_{L,\varepsilon}}), b_L^\varepsilon(\cdot, z_{n_{L,\varepsilon}}) \rangle \end{pmatrix}$$

be the $n_{L,\varepsilon} \times n_{L,\varepsilon}$ corresponding Gramian matrix. This matrix defines a uniformly bounded operator in ℓ^2 which is also bounded below uniformly.

It is clear that one can apply the estimate (2) to the Bochner-Riesz type kernel with coefficients given by the function $\beta_\varepsilon^2(x)$ getting

$$(4) \quad |\Delta_{ij}| \sim \frac{1}{L^m} \left| \int_M B_L^\varepsilon(z, z_i) \overline{B_L^\varepsilon(z, z_j)} dV(z) \right| \lesssim \frac{1}{(1 + Ld(z_i, z_j))^N}.$$

To define our uniformly bounded functions we will use the entries of matrix $\Delta^{-1/2}$. The following localization result by Jaffard, [7, Proposition 3], say basically that if we have an invertible matrix in ℓ^2 which is well localized on the diagonal, its inverse matrix is also localized along the diagonal and thus bounded in ℓ^p .

Theorem 2.10 (Jaffard). *Let (X, d) be a metric space such that for all $\epsilon > 0$ there exists C_ϵ with*

$$\sup_{s \in X} \sum_{t \in X} \exp(-\epsilon d(s, t)) \leq C_\epsilon,$$

and that for a given $N > 0$

$$\sup_{s \in X} \sum_{t \in X} \frac{1}{(1 + d(s, t))^N} = B < \infty.$$

Let $A = (A(s, t))_{s, t \in X}$ be a matrix with entries indexed by X and such that for $\alpha > N$

$$(5) \quad |A(s, t)| \leq \frac{C}{(1 + d(s, t))^\alpha}.$$

Then, if A is invertible as an operator in ℓ^2 the entries of the matrix A^{-1} (and also $A^{-1/2}$ when A is positive definite) satisfies the same kind of bound (5), and therefore the operators defined by these matrices are bounded in ℓ^p for $1 \leq p \leq \infty$, with bounds depending only on the constants C_ϵ, B and C .

The following proposition allow us to apply Jaffard's result to the matrix Δ and it will be used also to bound our orthonormal functions. We observe that it is precisely in this point where we need estimate (2). As we mentioned before, the reproducing kernel can be bounded with the same bound than in (2) but just with $N = 1$.

Proposition 2.11. *Let $\{z_j\} \subset M$ be uniformly separated and let $N > m$. Then*

$$\sup_{z \in M} \sum_j \frac{1}{(1 + Ld(z, z_j))^N} \lesssim 1.$$

Proof. Let $\delta > 0$ be the separation. Assume that $d(z, z_j) \geq \delta/(L + 1)$ for all j . Then $d(z, w) \leq \frac{3}{2}d(z, z_j)$ for $w \in B_j = \{w \in M : d(z_j, w) < \delta/2(L + 1)\}$. Therefore

$$\begin{aligned} \sum_j \frac{1}{(1 + Ld(z, z_j))^N} &\lesssim L^m \sum_j \int_{B_j} \frac{1}{(1 + Ld(z, w))^N} dV(w) \\ &\lesssim L^m \int_M \frac{1}{(1 + Ld(z, w))^N} dV(w) = L^m \int_0^1 V(\{w \in M : (1 + Ld(z, w))^{-N} > t\}) dt. \end{aligned}$$

Using that for geodesic balls $V(B(z, r)) \sim r^m$ one can bound the integral above by a constant times $L^{-m} \int_0^1 t^{-m/N} dt$. The case when $d(z, z_j) < \delta/(L + 1)$ for some j follows easily. \square

We apply Jaffard's result to the matrix Δ considering the distance $d(i, j) = Ld(z_i, z_j)$ in $X = \{1, \dots, n_{L, \epsilon}\}$ and the estimate (4). The two required properties in Theorem 2.10 can be easily deduced as in Proposition 2.11 so we get

$$|\Delta_{ij}^{-1/2}| \lesssim \frac{1}{(1 + Ld(z_i, z_j))^N},$$

and

$$\|\Delta^{-1/2}\|_{\ell^\infty \rightarrow \ell^\infty} = \max_i \sum_j |\Delta_{ij}^{-1/2}| \lesssim 1.$$

To define the orthonormal functions we follow [16]. Denote $\Delta_{ij}^{-1/2} = B_{ij}$ and define the orthonormal set of functions from E_L

$$\Psi_i^L = \sum_j B_{ij} b_L^\varepsilon(\cdot, z_j).$$

And then the functions from E_L

$$s_i^L = \frac{1}{\sqrt{n_{L,\varepsilon}}} \sum_j \zeta^{ji} \Psi_j^L,$$

where $\zeta = e^{2\pi i/n_{L,\varepsilon}}$. They are orthonormal because

$$\langle s_i^L, s_k^L \rangle = \frac{1}{n_{L,\varepsilon}} \sum_{j=1}^{n_{L,\varepsilon}} \zeta^{j(i-k)} = \delta_{ik}, \quad 1 \leq i, k \leq n_{L,\varepsilon}.$$

To verify that the s_i^L are indeed uniformly bounded we define the linear maps

$$F_L : \mathbb{C}^{n_{L,\varepsilon}} \longrightarrow E_L, \quad v = (v_i) \mapsto \sum_j v_j b_L^\varepsilon(\cdot, z_j).$$

Again, by Proposition 2.11, these maps have ℓ^∞ to $L^\infty(M)$ norm bounded by

$$\sup_{z \in M} \sum_j |b_L^\varepsilon(z, z_j)| \lesssim L^{m/2} \sup_{z \in M} \sum_j \frac{1}{(1 + Ld(z, z_j))^N} \lesssim L^{m/2}.$$

So, finally we get

$$\|s_i^L\|_{L^\infty(M)} \leq \frac{1}{\sqrt{n_{L,\varepsilon}}} \|F_L\|_{\ell^\infty \rightarrow L^\infty(M)} \|\Delta^{-1/2}\|_{\ell^\infty \rightarrow \ell^\infty} \lesssim 1,$$

for all $L \in \mathbb{Z}^+$ and $1 \leq i \leq n_{L,\varepsilon}$.

REFERENCES

- [1] A. ALEXANDROV, *Inner functions on compact spaces*, Functional. Anal. Appl. 18 (1984), 87-98.
- [2] E. BANNAI, R. M. DAMERELL, *Tight spherical designs*, I. J. Math. Soc. Japan 31, no. 1, 199-207, 1979.
- [3] J. BOURGAIN, *Applications of the spaces of homogeneous polynomials to some problems on the ball algebra*, Proc. Amer. Math. Soc. 93 (1985), no. 2, 277-283.
- [4] O. CHRISTENSEN, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis. Birkhuser Boston, Inc., Boston, MA, 2003.
- [5] F. FILBIR, H. N. MHASKAR, *A quadrature formula for diffusion polynomials corresponding to a generalized heat kernel*, J. Fourier Anal. Appl. 16 (2010), no. 5, 629-657.
- [6] L. HÖRMANDER, *The spectral function of an elliptic operator*, Acta Math. 121 o (1968), 193-218.
- [7] S. JAFFARD, *Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications.* (French) [Properties of matrices "well localized" near their diagonal and some applications] Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990), no. 5, 461-476.
- [8] N. LEV, J. ORTEGA-CERDÀ, *Equidistribution estimates for Fekete points on complex manifolds*, J. Eur. Math. Soc., to appear. (2013) arXiv:1210.8059 [math.CV].
- [9] J. MARZO, *Marcinkiewicz-Zygmund inequalities and interpolation by spherical harmonics*, J. Funct. Anal. 250 (2007), no. 2, 559-87.

- [10] J. MARZO, J. ORTEGA-CERDÀ, *Equidistribution of Fekete points on the sphere*, Constr. Approx. 32 (2010), no. 3, 513-521.
- [11] A. OLEVSKII, *Fourier series with respect to general orthogonal systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 86. Springer-Verlag, New York-Heidelberg, 1975.
- [12] J. ORTEGA-CERDÀ, B. PRIDHNANI, *Carleson measures and Logvinenko-Sereda sets on compact manifolds*, Forum Math. 25 (2011), 1, 151-172.
- [13] J. ORTEGA-CERDÀ, B. PRIDHNANI, *Beurling-Landau's density on compact manifolds*, J. Funct. Anal. 263 (2012) 2102-2140.
- [14] W. RUDIN, *New constructions of functions holomorphic in the unit ball of \mathbb{C}^n* , CBMS Regional Conference Series in Mathematics, 63, AMS, Providence, RI, 1986.
- [15] J. RYLL and P. WOJTASZCZYK, *On homogeneous polynomials on a complex ball*, Trans. Amer. Math. Soc. 276 (1983), 107-116.
- [16] B. SHIFFMAN, *Uniformly bounded orthonormal sections of positive line bundles on complex manifolds* (2014), arXiv:1404.1508 [math.CV].

DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI
UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08007-BARCELONA, SPAIN
E-mail address: jmarzo@ub.edu

DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI
UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08007-BARCELONA, SPAIN
E-mail address: jortega@ub.edu